Volume: 1, Issue: 4
Page: 72 - 92
2019

ISSN 2664-7354

International Journal of Academic Research in Business, Arts and Science (IJARBAS.COM)

On Mathematical Models for Pension Fund Optimal Selection Strategies

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Abstract:

Pension, being regular payments made to retirees or their beneficiaries after retiring from active service, needs efficient and effective management because of the funds involved as the living standard of the retirees and their dependents rest on it after retirement. In attempt to maximize the wealth of pension contributors, the investors may end up losing the pension fund assets because higher returns on investment go hand in hand with higher risk of loss of invested contributions/savings. This can shatter the hope of not just the contributors/investors but also the entire country as a whole. This research has developed and modified the Dynamic Accumulation Model (DAM) and Risk Minimizing Model (RMM) to aid the optimal fund selection among the four (4) types of fund available in Contributory Pension Scheme (CPS) in Nigeria in order to solve the problem of safety or uncertainty of invested amount. The models strike a balance between the objectives of wealth maximization and risk minimization of pension fund investment in Nigeria.

IJARBAS
Accepted 15 December 2019
Published 10 January 2020
DOI: 10.5281/zenodo.3603511

Keywords: wealth maximization, models, contributions/savings, optimal pension fund selection, risk minimization,

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1.0 BACKGROUND OF THE STUDY.

Older persons are a particularly vulnerable group of people due to a decline in their physical, mental and consequently economic powers. To stem this tide, pension scheme was introduced by the government. Pension is a series of periodic money payments made to a person who has retired from employment because of age, disability/health or the completion of an agreed span of service [1]. The purpose of pension scheme is to provide employees regular and stable income after their retirement from active service. It is an arrangement an employer or a group of employers use to provide pension benefits for their employees when they leave or retire from service. The pension scheme in Nigeria is funded by contributions from just the employer or from both the employer and the employees [2]. Workers in the public and private sectors (with 15 or more employees) shall be entitled to pension upon retirement [3]. The contributions for any employee to which the Nigerian Pension Reform Act (PRA) applies shall be a minimum of 10% by the employer and 8% by the employee. An employer may also agree to bear the full responsibility of the contribution provided it is not less than 20% of the monthly emolument of the employee. A good pension plan should not only serve as an incentive for employees but also helps the employer retain/attract experienced members of staff. The management of pension in Nigeria is inundated by multiple and diverse problems. The contributors/investors are faced with the problem of determining appropriate investment portfolios in order to ensure safety of the investible funds and provide adequate safeguard for pension fund assets. Also, the inability of the pension operators to utilize the available pension funds is worrisome, as many public organizations find it extremely difficult to secure money to pay entitlements of their retirees and pensioners due to wrong choice of investment strategy. Therefore, the objective of this research is to develop and modify conceptual mathematical models for pensioners' wealth maximization and investment risk minimization of stochastic outcomes. The regulations on investment of pension fund assets are focused at deepening the pension market, particularly by enlarging the list of allowable investment instruments open to pension fund operators. Conversely, this makes the market to be open to more risks given the sensitivity of the pension assets and the volatility of the instruments. This study will focus on how to strike an appropriate balance between the profitable investments on one hand, and the security of investible assets on the other. Prior to 2004, the Nigerian pension industry under the old defined benefit pension system (in which the retirement benefits were fixed) had a deficit of over ₦2 trillion [4]. Since the 2004 reform introduced the Contributory Pension Scheme (CPS), the industry has witnessed exponential growth. The Pension Reform Act 2014 repealed the 2004 Pension Reform Act. It governs and regulates the administration and management of the uniform CPS for both the public and the private sectors in Nigeria. As at the end of September

2015, total pension contribution in the custody of Pension Fund Custodians (PFCs) and under the management of Pension Fund Administrators (PFAs), including closed PFAs was in excess of ₦4.8 trillion [4]. However, this figure was only 5% of the country $510 billion GDP compared to 170% in Netherland, 131% in Britain and 113% in the US. Moreover, 6.5 million out of 80 million working population enrolled into the CPS. A pension fund is an investment pool into which scheme pays contributions to build up a lump sum for providing income in retirement. The huge investible fund from this pool is a potential game changer and a key growth driver if securely and profitably invested. The main reason for the reform of pension system in Nigeria was to establish a uniform set of rules, regulations and standards for the administration and payment of retirement benefits for public and private sectors. It provides for smooth operations of the CPS; ensures that every person who worked in either public/private sector receives retirement benefits as and when due; and assists improvident individuals by ensuring they save in order to cater for their livelihood during old age. The reform has been designed to lower the burden on a shrinking number of workers and reduce strain on public budget. It has also strengthened the link between pension contributions and benefits by prolonging the contribution value through raising the retirement age and diversification of sources of retirement pension benefits. The most favoured approach has been the replacement of the Pay-As-You-Go (PAYG) System with a fully funded system so that retirement income will be fully financed by investing the pension plan members' contributions. In Nigeria, the investments of pension fund assets are regulated by the National Pension Commission (PenCom) as provided under Part XII of Section 85-91 of the Pension Reform Act (PRA) 2014. Section 85(1) stipulates that all funds realized under the CPS shall be invested by the PFA with the objectives of safety and maintenance of fair returns on the amount invested. Section 86 lists the modes of investment of pension fund, subject to PenCom regulations to include:

- bonds, bills and other securities issued or guaranteed by the Central Bank of Nigeria (CBN) and the Federal, the State and the Local Governments;
- bonds, debentures, redeemable preference shares and other debt instruments issued by corporate entities listed on a stock exchange registered under the Investment and Securities Act (ISA);
- ordinary shares of public limited companies listed on a securities exchange commission registered under ISA;
- bank deposits and bank securities;
- investment certificates of closed-end investment funds or hybrid investment funds listed on a securities exchange commission registered under ISA with good track records of earnings;

- units sold by open-end investment funds or specialist open-end investment funds registered under ISA;
- real estate development investments or specialist investment funds; and
- such other financial instruments as PenCom may, from time to time, approve.

Section 87 permits a PFA to invest pension funds in units of any investment outside Nigeria within the categories of approved portfolios, subject to the President's approval, subsisting foreign exchange rules of the CBN and the portfolio limits for investment of pension fund assets outside Nigeria. However, Sections 88-90 place some restrictions on certain areas of investment. Furthermore, Section 90 states that the scope of the restrictions on some areas of investment or allowable instruments may be widened or altered through regulations or guidelines made by PenCom. The pension funds differ in their risk profiles. The Risk Management Committee established by a PFA determines the risk profile of investment portfolios of the PFA. Multi-fund structure has also been introduced to provide for different categories into which pension funds can be divided for investment purposes. A contributor is also given a possibility to choose the fund wished to invest in and allowed to change choice periodically based on some factors which may include age of the contributor, work status, risk exposure elements and so on. The returns on the investment of pension funds have stochastic characteristics as the investments are more or less risky. Two types of models will be developed and modified in this study. They will give the contributors a proposal on which fund is optimal for them to choose at each rebalancing period during their active life depending on certain parameters. Moreover, apart from the context of pension, the approach of this research can also be applied to other areas involving investment of money with different and periodic characteristics. According to PenCom, there are rules and regulations governing the ways in which the PFAs are to invest the pension fund assets under the management and custody of PFCs [5]. The Regulation, in its Section 7 (multi-fund structure), classifies the pension funds managed by the PFAs into four (4) types of funds based on the overall exposure to variable income instruments.

TABLE 1: CLASSIFICATIONS OF PENSION FUND TYPES.

FUND TYPES	CONTRIBUTORS/INVESTORS	THRESHOLDS OF PORTFOLIO VALUE
1	Strictly by formal request from a contributor and suitable for contributors who want to invest in high risk projects with higher rewards.	75% maximum 20% minimum

2	Active contributors who are 49 years and below as at their last birthdays.	55% maximum 10% minimum
3	Active contributors who are 50 years and above as at their last birthdays.	20% maximum 5% minimum
4	Exclusive for retirees	10% maximum 0% minimum

Source: The Nigerian National Pension Commission (PenCom), 2017.

2.0 DECISION PROBLEMS.

Subsequent to the implementation of the multi-fund structure, the participants choose the type of fund they desire to be assigned and are allowed to revise their decision during the period of savings and switch to another fund eventually. Hence, future pensioners are able to partially influence the amount of their savings at retirement time and also the risk of balancing between the four fund types, subject to additional government regulatory restrictions imposed on the fund selection [5]. A detailed study of the regulations and restrictions is contained in [6]. Assuming that the workers' expected retirement time is in T years and they save for their pension in pension fund management institution offering investment in funds labeled $1, 2,\ldots,J$. It can also be assured, without loss of generality, that they revise their decision every twelve months (1 year). It is now easy to formulate this problem. Time $T \in \{0,1,2,\ldots,T\text{-}1\}$ can be used to determine the fund $J_t \in \{1,2,\ldots,J\}$ so that the best possible outcome at time T can be obtained. Since the fund is invested in instruments with more or less volatile returns, the outcome will be stochastic. It is therefore necessary to introduce a measure that gives means for comparing two random outcomes and determining the better one. Two approaches can be used for this purpose: the expected utility and the risk measure concepts. Therefore, the problem formulated earlier can be approached in two different ways: maximizing the expected utility from the saved amount at time T; and minimizing the riskiness of the savings (investment) at time T. Let d_T be a random variable representing the saved contributions at time T; U be the contributor's utility function; and R be the corresponding risk aversion coefficient fixed at the value $\hat{R} \in R$; J be a set representing eventual restrictions on the fund selection imposed by the government and other constraints that may come into consideration; d_t be the state variables representing the contributions at time t; and X the set representing constraints on d_t. The first (maximization) approach can then be formulated as the following optimization problem:

$$\max_{j_t,\, t \in \{0,1,2,\ldots,\, T-1\}} \quad E(U(d_T))$$

Subject to:

$$R = \hat{R}$$

$$d_t \in X$$
$$J_t \in J \qquad ...(1)$$

The above problem leads to a stochastic dynamic programming problem. The second (minimization) approach uses the notion of risk measures which are usually statistical tools suitable for qualifying and quantifying the riskiness of a future outcome. In this research, a single period or static risk measure will be used. If M is the used risk measure and μ is the target amount, then the problem to be solved is:

$$\underset{j_t, t \in \{0,1,2,\ldots, T-1\}}{Min} \quad M(d_T)$$

Subject to:

$$E(d_T) \geq \mu$$
$$d_t \in X$$
$$J_t \in J \qquad ...(2)$$

Average value-at-risk deviation will be applied as the risk measure. Before going into the theoretical framework of risk measures, the following questions are important for a contributor/investor:

- What is the risk tolerance or risk aversion level?
- What amount is expected to be achieved?

2.1 UTILITY FUNCTIONS.

The answers to the questions asked earlier specify the utility function U defined in *(1)*, and the risk measure M defined in *(2)*. If $U(x)$ is a utility function representing one's preference and $f{:}R \to R$ is an increasing function, then $f(U(x)) \geq f(U(y))$ if and only if $U(x) \geq U(y)$. Investors often face the necessity of making decisions about investment, the efficiency or return which depends on unknown future behavior of some stochastic environment. The Expected Utility Theory states that the decision makers choose between risky and uncertain prospects by comparing their expected utility values. If X is a random variable, then the expected utility associated with X is $E(U(x))$ where E is the expectation operator. In the context of pension funds, let d denotes the random variable representing future pensioners' contributions in their retirement saving accounts (RSAs). If the random wealth d depends on other stochastic or deterministic factors and a decision variable j where J is the set of all feasible decisions j, the future pensioners solve the problem:

$$\underset{j \in J}{Max} \quad E(U(d_j))$$

The most important thing here is the right choice of the utility function and its parameters, reflecting in particular investors' attitude to risk. Based on the attitude to risk, the utility functions of risk averse, risk neutral and risk loving investors will be concave, affine and convex respectively. A risk aversion coefficient is a special measure reflecting the characteristic and degree of investors' risk aversion. Intuitively, the more concave the expected utility function, the more risk averse the investors. Risk aversion can be measured by the second derivative of the utility function, $U''(x)$. The risk aversion coefficient can be absolute, constant or relative[7]. The absolute risk aversion coefficient, $\lambda_A(x)$, at a point x pertaining to a utility function U is defined as:

$$\lambda_A(x) = \frac{U''(x)}{U'(x)} \qquad ...(3)$$

where $U'(x)$ and $U''(x)$ are the first and second derivatives of the utility function respectively. Utility function with a constant absolute risk aversion coefficient is called CARA (constant absolute risk aversion) Utility Function. This is used when the absolute risk aversion coefficient does not depend on the wealth (that is, $\lambda'_A(x) = 0$, where $\lambda'_A(x)$ is the first derivative of $\lambda_A(x)$). It is observed that there is natural assumption that most investors have decreasing absolute risk aversion (DARA). U exhibits DARA if richer contributors are less absolutely risk averse than poorer ones (that is, $\lambda'_A(x) < 0$). The increasing absolute risk aversion (IARA) is the opposite case of DARA. The relative risk aversion coefficient, $\lambda_R(x)$, at a point x pertaining to a utility function U is defined as:

$$\lambda_R(x) = -x\frac{U''(x)}{U'(x)} \qquad ...(4)$$

Therefore, utility function with a constant relative risk aversion is called CRRA Utility Function. There are several classes of utility functions suitable for describing various types of investors' behavior. The well-known classes are quadratic, exponential and power-like utility functions[8]. In this research, power-like utility function will be used to model the expected utility maximization for pension funds. A power-like utility function is of the form;

$$U(x) = \frac{x^{1-a}}{1-a}$$

Additional contributions can be made into RSA if the proposed pensioners decide to cut down their current expenses or consumption while in active service. The reciprocal of a, (i.e $\frac{1}{a}$), is the ratio of the inter-temporal substitution between consumption and savings at a particular period of time. That is, it measures the willingness to substitute consumption with additional pension fund contributions. The smaller the value of a, the larger the value of $\frac{1}{a}$ and the more willing the contributors are able to substitute consumption over time. Note that a is the coefficient of relative risk aversion defined by *(4).* Since the coefficient of the relative risk aversion is constant, this utility function is a CRRA (constant relative risk aversion) function.

The expression x^{1-a} is increasing in x if $a<1$ but decreasing if $a>1$. Dividing the expression by $1-a$ ensures that the marginal utility is positive for all values of a.

2.2 RISK MEASURES.

Measures of risk are functions that describe risk and give the pension fund managers or financial decision makers a quantitative tool to compare different insecure alternatives. Although there are many ways financial risk can be measured, this study will focus on the value-at-risk deviation and the average value-at-risk deviation because they are often used risk measures and have better properties than others which treat the negative and positive deviations from the mean in the same way. There are three types of functional connected to the theory of risk measures: Acceptability (A), Risk Capital (E) and Deviation Risk (D).

$$A(Y)=E(Y) - D(Y).$$

Rockafellar et al in [9] use the notions of sureness valuations, expectation-bounded risk measures and general deviation measures instead of acceptability, risk capital and deviation risk functional respectively. Given a probability distribution of future wealth of a contributor, the value-at-risk (expected shortfall) at the confidence level α of the future wealth random variable, is a maximum wealth exceeded with probability $1-\alpha$. When this risk measure is used, we accept positions as safe if in less than $\alpha\%$ of the cases we experience difficulties. The value-at-risk $VaR_\alpha(Y)$ of a profit random variable Y with a distribution F at a confidence level α, where $0<\alpha<1$, is defined as the α-quintile $F^{-1}(\alpha)$. The value-at-risk deviation of a profit random Y at a confidence level α is defined by:

$$VaRD_\alpha(Y)=E(Y) - VaR_\alpha(Y) \qquad 0<\alpha<1$$

Let Y be a profit random variable with a distribution F, and F_α be the lower α-tail distribution which equals to 1 for profit exceeding VaR_α and equals $\frac{F}{\alpha}$ for profits below or equal to VaR_α. The average value-at-risk of Y at the level α is defined as the mean of the α-tail distribution F_α. If Y is a continuous random variable, the average value-at-risk of Y at level α, $0<\alpha<1$, is defined as:

$$AVaR_\alpha(Y)=\frac{1}{\alpha}\int_0^\alpha F^{-1}(u)du \qquad ...(5)$$

where F is the distribution of Y. Therefore, the average value-at-risk deviation is defined by:

$$AVaRD_\alpha(Y)=E(Y) - AVaR_\alpha(Y) \qquad ...(6)$$

When investors want to construct a portfolio from assets, they aim to maximize the portfolio returns. Risk averse investors minimize the risk associated with the investment as well. This problem has not a unique solution in general. One has to find a compromise point between the return and the risk. The curve comprising all the optimal solutions (i.e, portfolios with maximal return and minimal risk) is called the Efficient Frontier[10]. In the pension planning models, the random future outcome is the amount d_T of money saved at year T of pension savings. The contribution is influenced by the following factors: the stochastic fund returns, the contributor's decision about the fund selection and the salary growth. If these factors are symbolically denoted by x, then $d_T = d_T(x)$. The standard decision problem is to maximize the acceptability of the outcome over all feasible decisions $x \in X$. Thus, the optimization problem, after taking $A = E - D$, is of the form:

$$\max_{x \in X} \quad E(d_T(x)) - D(d_T(x)) \qquad \ldots(7)$$

Using μ as a parameter, solving problem *(7)* for an appropriate range of μ leads to the efficient frontier function in *(8)* pertaining to the functional *D.*

$$\mu \longmapsto F(\mu) = \min\{ D(d_T(x)) : E(d_T(x)) \geq \mu, x \in X \} \qquad \ldots(8)$$

3.0 MATHEMATICAL MODELS FOR PENSION FUND INVESTMENT.

Constructing models suitable for solving pension problems stated in the previous sections helps to find optimal strategy between pension fund types with different risk profiles in a time horizon of T years. In pension saving, one should take into account the future contributions. If a series of contributions throughout a working life span is made, a fall in the assets value early in working life span does not affect the whole contributions. That is, only part of one's future pension wealth will be affected. On the other hand, if it occurs close to retirement, it will affect all past accumulated pension wealth (that is, accumulated contributions and returns on them). Therefore, it is reasonable that the investment decision depends on the time to the maturity of savings. In other words, investors with a long time horizon can prefer to invest in risky assets than contributors with imminent retirement years. For instance, contributions of an employee who has more years to retire can be invested in stocks while that of an employee who has less years to retire can be preferably invested in bonds. Pension saving becomes more conservative as retirement approaches in order to guarantee at least a minimum living standard in retirement. For solving the problems of optimal fund selection in pension planning, two types of models will be proposed in this study. They are: Dynamic Accumulation Model (DAM) and Terminal Risk Minimization Model (TRMM). Since retiring persons strives to maintain the living standard at the same level as their last pre-retirement income, the wealth at year t is measured by the multiples of t-year's salary instead of the absolute value of the saved amount. DAM deals with the maximization of the expected wealth or utility of the saved amount while TRMM deals with the minimization of the riskiness of the investment decision taken. TRMM uses a static risk measure to minimize the uncertainty of achieving the target wealth[11].

3.1 MAXIMIZATION MODEL.

In Dynamic Accumulation Model (DAM), we will assume: yearly (annual) rebalancing, the contributor's utility function U is known and the attitude to risk represented by risk aversion coefficient is also known. Therefore, the expected utility is maximized from the terminal wealth[12]. Before proceeding to the problem formulation, the following notations to be used need to be clarified.

T	expected retirement time,
J	number of funds,
r_t^j	returns on fund j at time t,
u_t	accumulated sum at time t (where $t\epsilon 0,1,\ldots, J$),
w_t	gross salary at time t,
β_t	salary growth at time t defined by $w_{t+1}= w_t(1+\beta_t)$,
d_t	ratio of accumulated sum u_t to the salary w_t
$\mathbb{T}$	rate of regular yearly contribution as a part of gross salary.

Suppose a future pensioner with the expected retirement time in T years deposits once a year a $\mathbb{T}$-part of the yearly salary w_t at year t to a fund $j\epsilon\{1,2,\ldots, J\}$. Since the funds invest in financial markets, the returns r_t^j are stochastic[13]. The start up value u_0 is equal to the first contribution (i.e, $u_0 = w_0\mathbb{T}$). At each next decision time $t=1,2,\ldots,T-1$, the amount u_t is appreciated by a return corresponding to the chosen fund j at the previous time stage $t-1$, and a new contribution is added to the account. Under the assumption of constant contribution rate $\mathbb{T}$, the equation describing the time evolution of the account is:

$$u_{t+1} = u_t(1 + r_t^j) + w_{t+1}\mathbb{T},\ \ t=0,1,\ldots,T-1 \qquad \ldots(9)$$

At the time of retirement T, the pensioners will strive to maintain their living standard at the level of their last salary. From this point of view, the saved contributions u_T at time T is not precisely what the future pensioner cares about. The ratio of the cumulative sum u_T and the yearly salary w_T (i.e,$\frac{u_T}{w_T}$) is more important. Using the quantity $d_t = \frac{u_T}{w_T}$, one can reformulate the constraint equation as:

$$d_{t+1} = F_t(d_t, j),\ \ t = 0,1,\ldots,T-1 \qquad \ldots(10)$$

where $\quad F_t(d_t, j) = d\frac{1 + r_t^j}{1+ \beta t} + \mathbb{T}, \qquad t = 0,1,\ldots,T-1$

and $\quad w_{t+1} = w_t(1+ \beta_t), \qquad t = 0,1,\ldots,T-1.$

The investor's decision about the fund selection at time t is based on the information at that time. If I_t denotes the information consisting of the history of returns $r_{t'}^j$, $t'=0,1,\ldots,t-1$, $J\epsilon\{1,2,\ldots,J\}$ and the wage growth $\beta_{t'}$, $t' = 0,1,\ldots,t-1$, until time t, then we have $j = j(t, I_t)$. At this point, we make two assumptions for DAM:

- The fund return r_t^j for all funds $j\epsilon\{1,2,\ldots, J\}$ and all time stages $t\epsilon\{0,1,\ldots,T\}$ are stochastic and mutually independent for fixed j.
- The wage growth rates β_t , $t = 0,1,\ldots,T-1$, are deterministic and prescribed.

Both assumptions imply that the quantity d_t is the only relevant information from I_t. Hence, $j(t, I_t) \equiv j(t, d_t)$. In order to maximize the contributor's utility from the terminal wealth, one can formulate a problem of stochastic dynamic programming with the constraint equation defined in *(10)* where the maximum;

$$\max_{J} \; E(U(d_T)) \qquad \ldots(11)$$

is taken over all non-anticipated strategies $J = \{ j(t, d_t) : t = 0,1,\ldots,T-1\}$. Here U denotes the preferred utility function of a contributor. According to [14], the optimal strategy of problems *(10)* & *(11)* is the solution of the Bellman equation:

$$v_t(d) = \max_{j \in \{1,2,\ldots, J\}} E[v_{t+1}(F_t(d, j, r_t^j))] = E[v_{t+1}(F_t(d, j(t, d), r_t^j))] \qquad \ldots(12)$$

where $t = 0,1,\ldots,T-1$ and $v(d)=u(d)$.

One can conclude that maximizing $E(U(d_T))$ is the same as maximizing the conditional expectation $E(U(d_T) \mid d_t)$ for arbitrary t, where:

$$E(U(d_T) \mid d_t) = E(E(U(d_T) \mid d_{t+1}) \mid d_t) \qquad \ldots(13)$$

Using *(13)*, one can obtain:

$$V_t(d) = \max_{j_t} \max_{j_{t+1}} E(E(U(d_T) \mid d_{t+1}) \mid d_t = d) \qquad \ldots(14)$$

If J_{t+1} is a strategy, then we denote $R_{t+1}(J_{t+1})$ as the sequence of fund returns determined by the strategy J_{t+1}. That is;

$$R_{t+1}(J_{t+1}) = \{ r_t^{j_{\mathbb{T}}}; \mathbb{T} = t+1,\ldots,T \text{ and } j_{\mathbb{T}} \text{ given in } J_{t+1}\}.$$

Recall $v_T(d)=u(d)$ and proceeding backward from $t = T-1$ down to $t = 0$, one can calculate the optimal strategy $j_t(d_t)$. The optimal strategy of *(12)* gives the investor the information about the optimal fund selection for each time t in dependency on the value of contributions d_t. Now, suppose that the stochastic fund returns r_t^j are represented by their densities f_t^j and $d\,\frac{1+r_t^j}{1+\beta t} + \mathbb{T}$ defined earlier is represented by y, then *(10)* & *(12)* can be rewritten in the form:

$$V_t(d) = \max_{j \in \Delta_t} E[v_{t+1}(F_t(d, j, r_t^j))] \qquad \ldots(15)$$

where $\Delta_t \subset \{1,\ldots,J\}$ represents the set of all funds that may be chosen by an investor at time t having taken into account the government restrictions imposed on such fund selection. Problem *(15)* can also be simplified further.

$$v_t(d) = \underset{j \in \Delta_t}{Max} \int_{\mathbb{R}} v_{t+1} \ (d\ \frac{1+r_t^j}{1+\beta t} + \mathbb{T}) f_t^j(r)\, dr.$$

$$= \underset{j \in \Delta_t}{Max} \int_{\mathbb{R}} v_{t+1}\ (y) f_t^j((y-\mathbb{T})\frac{1+\beta t}{d} - 1)\frac{1+\beta t}{d} dy$$

$$= \int_{\mathbb{R}} v_{t+1}\ (y) f_t^{j(t,\ d)}((y-\mathbb{T})\frac{1+\beta t}{d} - 1)\frac{1+\beta t}{d}\, dy \qquad ...(16)$$

Where $\mathbb{R}$ denotes the set of real numbers.
(16) is the mathematical formulation whose aim is to determine optimal j where $V_T(d)=U(d)$ and f_t^j is the density function of normally distributed fund returns r_t^j. We use the constant relative risk aversion utility function of the form:

$$U(d) = -d^{1-a}, \ d>0 \qquad ...(17)$$

Where $a>1$ is the constant coefficient of relative risk aversion. We note that the function $U(d)$ defined by *(17)* is a smooth, increasing and strictly concave function for $d>0$. The coefficient of relative risk aversion a is commonly suggested to be less than *10*[15]. The principal difficulties in computing the integral *(16)* is due to significant oscillations in the integrand function. Moreover, it may attain large values as well as low values. Therefore, a scaling technique is needed when computing the integral. The idea of scaling is rather standard and is widely used in similar circumstances. Let $H_t(d)$ be any bounded positive function for $t=1,2,\dots T$. We scale the function V_t by H_t. That is, we define a new auxiliary function.

$$W_t(d) = H_t(d)\ V_t(d).$$

The original function $V_t(d)$ can be easily calculated from $W_t(d)$ as follows:

$$V_t(d) = \frac{W_{t(d)}}{H_{t(d)}}$$

For each time step t from $t=T$ down to $t=1$, we have:

$$W_t(d) = \mathrm{H_T}(d)\ V_T(d)$$

$$W_{t-1}(d) = \mathrm{H_{t-1}}(d)\ V_{t-1}(d)$$

$$= \underset{j \in \Delta_t}{Max} \int_{\mathbb{R}} H_{t-1}(d) V_t(\frac{d}{1+\rho_t}(1+r) + \mathbb{T}) f_t^j(r)\, dr$$

$$= \underset{j \in \Delta_t}{Max} \int_{\mathbb{R}} \frac{H_{t-1(d)} W_{t(y)}}{H_{t(y)}} f_t^j((y-\mathbb{T})\frac{1+\rho_t}{d} - 1)\frac{1+\rho_t}{d} dy \qquad ...(18)$$

3.2 MINIMIZATION MODEL.

The risk minimizing model for solving the problem of pension fund investment strategies is based on minimizing the uncertainty or safety of the pension fund investment or contributed amount. The wealth is a random variable as it depends on the random returns of pension funds invested into various financial instruments. We look for an optimal selection of pension funds so that the target wealth is achieved in the sense of the average value while the uncertainty of the contributions/savings is reduced to the barest minimum using a static risk measure. We start with a description of the natural constraints that are to be taken into consideration/account. In the Dynamic Accumulation Model, they were given in *(10).* They express the appreciation of savings between two time stages and the regular contributions to the retirement savings account. We recall that $t \in \{0,1,\ldots,T-1\}$ denotes the time at which an investor makes a decision about the fund selection and $j \in \{1,2,\ldots,J\}$ denotes different funds with returns r_t^j at time t. Also the variable d_t denotes the ratio of the accumulated amount at time t to the salary at that time. We will now introduce other new variables for the Terminal Risk Minimizing Model (TRMM).

$y_t^1,\ldots,y_t^J$ — denotes the amount invested at time t in funds $1,\ldots,J$ respectively,

y_t — denotes the vector $[y_t^1, \ldots, y_t^J]^{\mathbb{T}}$ where d_t (from DAM) $= y_t^{\mathbb{T}}1 = \sum_{j=1}^{J} y_t^j$, 1 is a vector with all elements equal to 1.

$s_t^j = \frac{1+r_t^j}{1+\beta_t}$ — denotes the return of the fund j in the time interval $[t-1, t]$ adjusted by the wage growth rate β_t as defined in DAM by the relationship $w_{t+1}=w_t(1+\beta_t)$. It is assumed to be deterministic.

s_t — denotes the vector $[s_t^1, \ldots, s_t^J]^{\mathbb{T}}$.

The problem is to find the optimal y_t^j for all t, j, at each time t in order to achieve the target wealth μ and minimize the uncertainty. The equations describing the time evolution of savings dependent on balancing between funds are;

$$
\begin{aligned}
&y_0^{\mathbb{T}}1 = \mathbb{T} \\
&y_t^{\mathbb{T}}1 = y_{t-1}^{\mathbb{T}}s_t + \mathbb{T} \quad \text{for all } t \in \{0,1,\ldots,T-1\} \qquad \ldots(19) \\
&y_T^{\mathbb{T}}1 = y_{T-1}^{\mathbb{T}}s_T \\
&y_t \geq 0, \text{ for all } t \in \{1,\ldots,T\}
\end{aligned}
$$

Explaining the equations, the initial saved money at $t = 0$, i.e. the value of $y_0^{\mathbb{T}}1$, is equal to the first contribution $\mathbb{T}$. At each time stage $t = 1,\ldots,T-1$, the amount y_{t-1}^j from the previous time stage is appreciated by a corresponding random adjusted fund return s_t^j and a new contribution $\mathbb{T}$ is added. Therefore, the overall accumulated amount at time t is $y_{t-1}^{\mathbb{T}}s_t + \mathbb{T}$. The amount has to be distributed by an investor into funds $j \in \{1,2,..., J\}$ in parts y_t^j for which *(19)* must hold. At the end of saving when no contribution $\mathbb{T}$ is added, the next constraint appearing in the problem is the requirement on the minimal target amount μ in terms of the yearly salary. If $y_T^{\mathbb{T}}1$ (= d_T) is the wealth random variable, then;

$$E(y_T^{\mathbb{T}} 1) \geq \mu \qquad ...(20)$$

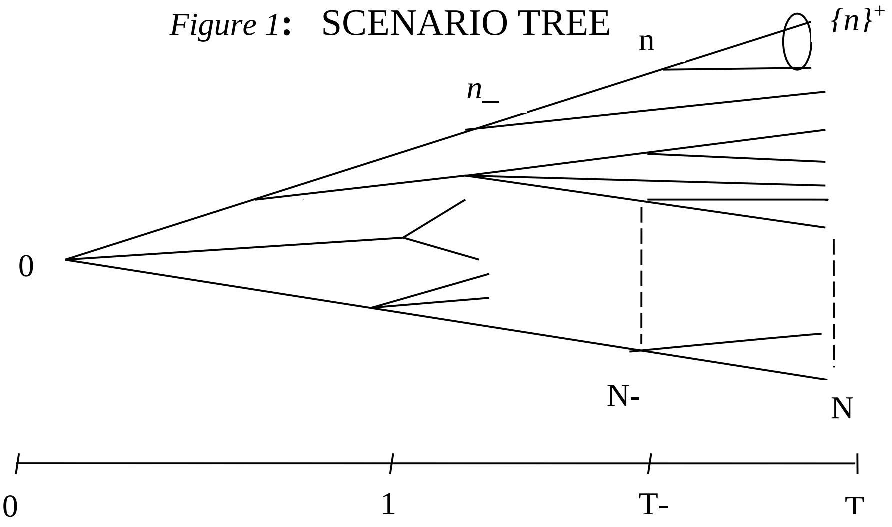
Hence, the objective function of the optimization problem is:

$$g(y) = E(y_T^{\mathbb{T}} 1) - AVaR_\alpha y_T^{\mathbb{T}} 1 \qquad ...(21)$$

This is minimized with respect to y_t^j, $t \in \{0,1,...,T\text{-}1\}$, $j \in \{1,2,..., J\}$, under constraints *(19) & (20).* The adjusted returns s_t will form a stochastic process in a discrete time[16]. Explaining by a scenario tree, each node at stage t of the scenario represents one possible state of the random vector s_t at the future time t. Each path in the tree, starting from the root means one scenario of evolution of the random process s_t we now denote:

0	the root of the tree.
$N=\{0,1,...,N\}$	the set of all nodes in the tree.
S	the number of terminal nodes in the tree,
$T=\{N\text{-}S+1,...,N\}$	the set of all S terminal nodes in the tree,
$N_o=\{I,...,N\text{-}S\}$	the set of inner nodes
$n\_$	the unique predecessor of the node $n \in N \setminus \{o\}$
$\{n\}^+$	the set of successors of the node $n \in N \setminus T$
$\varepsilon\{n\} \in \{0,...,T\}$	time stage of the node $n \in N.$

Figure 1: SCENARIO TREE



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In the scenario tree, the bottom line represents the time line. Each node *n*, except the root *n=0*, has exactly one predecessor *n_*. However, each node $n \in N \backslash T$ has a set of successors $\{n\}^+$. If lower index of variables y_t^j, r_t^j and s_t^j denote the particular node *n* in the time stage *t*, then we use notations y_n^j, r_n^j *and* s_n^j instead. Therefore, $y_n = [y_n^1, \dots, y_n^J]^{\mathbb{T}}$, $d_n = y_n^{\mathbb{T}} 1$ and $y_n \geq 0$ for all *n, j*. β_t changes to $\beta_{\xi\{n\}}$. It denotes the salary growth corresponding to time stage $\xi\{n\}$ of the node *n*. $s_n^j = \frac{1 + r_n^j}{1 + \beta_{\xi\{n\}}}$ represents the adjusted returns for all $n \in N \backslash \{o\}$, $j \in \{1, \dots, J\}$. s_n^j of fund *j* is valid in the period $\xi\{n\_\}$ to $\xi\{n\}$ with the corresponding scenario path between nodes *n_* and *n*. The wealth random variable d_T is represented by a vector of discrete values $d_m = y_m^{\mathbb{T}} 1$, $m \in T$ with the corresponding scenario probability $p_m > 0$, $\sum_{m \in T} p_m = 1$. The sum of node probabilities, $p_m > 0$, in every time stage *t* of the tree is equal to one. The minimized objective function in *(19)* can be expressed in the notation as :

$$\underset{y}{Min} \left[\sum_{m \in T} p_m (y_m^{\mathbb{T}} 1) = \underset{a}{Max} \left\{ a - \frac{1}{\alpha} \sum_{m \in T} p_m (y_m^{\mathbb{T}} 1 - a)^{-} \right\} \right] \qquad ...(22)$$

Subject to constraints *(17) & (18)* which can be rewritten as :

$$y_n^{\mathbb{T}} 1 = y_{n\_}^{\mathbb{T}} s_n + \mathbb{T} \text{ for all } n \in N_o \qquad ...(23)$$

$$\sum_{m \in T} p_m (y_m^{\mathbb{T}} 1) \geq \mu \qquad ...(24)$$

The optimization problems *(20)* to *(22)* are equivalent to the following:

$$\underset{a,z,y}{Min} \left[\sum_{m \in T} p_m (y_m^{\mathbb{T}} 1) - a + \frac{1}{\alpha} \sum_{m \in T} p_m z_m - N + S \right] \qquad ...(25)$$

subject to:

$$-a + y_m^{\mathbb{T}} 1 + z_m - N + S \geq 0, z_m - N + S \geq 0, \textit{for all } m \in T \qquad ...(26)$$

$$\sum_{m \in T} p_m (y_m^{\mathbb{T}} 1) \geq \mu \qquad ...(27)$$

$$y_n^{\mathbb{T}} 1 = y_{n\_}^{\mathbb{T}} s_n \textit{ for all } n \in T \qquad ...(28)$$

It is observed that every optimal solution for problems *(25)* to *(28)* is optimal for *(22)* to *(24)*. In other words, for every optimal solution to *(22)* to *(24)*, there exists optimal solution to *(25)* to *(27)*. A problem (25) to (28) is a linear program that can be symbolically written as:

$$\underset{x}{Min} \ c^{\mathbb{T}} x \qquad ...(29)$$

Subject to:

$$A_{ineq} x \leq b_{ineq} \qquad ...(30)$$

$$A_{eq} x = b_{eq} \qquad ...(31)$$

$$Z_m \geq 0 \text{ for all } m \in \{1, \dots S\} \qquad ...(32)$$

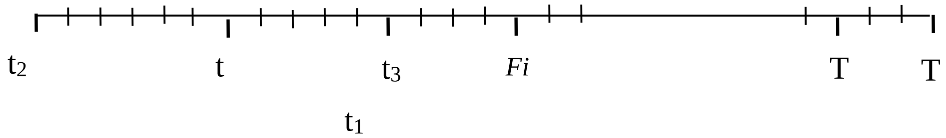
Where : A_{inq}= inequality constraint matrix,
A_{eq} = equality constraint matrix.

OBSERVATION AND TEST OF FEASIBILITY.

The vector of variables $x= (a, z, y)$ has the length $vars = 1 + S + J(1 + N)$. The matrix A_{ineq} is of type $(1 + S) \times vars$ and it is sparse with $(2J + 2)S$ nonzero elements. The sparse $(1+N) \times vars$ matrix A_{eq} has $(1 + 2N)J$ nonzero elements. It is now necessary to investigate the feasibility and the optimality of problems *(29)* to *(32)*. It is observed that the constraint *(31)* has no real impact on feasibility of the problem as it only describes explicitly the wealth evolution along the constraint. Similarly, constraint *(30)* does not influence the feasibility of *(29)* to *(32)*, with exception of one equality. Given any y_m, $m \in T$, we may put $z_m - N+S \geq 0$ arbitrary $a \leq y_m^{\mathbb{T}} 1 + z_m - N+S$. Hence, given any y, we may easily find feasible values of the variables a and z. However, we may treat the scenario s_n^j as fixed and given, the constraint $-\sum_{m \in T} p_m (y_m^{\mathbb{T}} 1) \leq -\mu$ is crucial in determining whether the problem is feasible or not. It is clear, given fixed scenario s_n^j and following *(31)*, it is not possible to achieve an arbitrary value of $E(y_T^{\mathbb{T}} 1)$. That is, it is not possible to achieve a too high outcome μ when the fund returns simulated in scenarios are low. Therefore, there exists a μ_{max} such that if $\mu > \mu_{max}$, the problems *(29)* to *(32)* are infeasible. The value is determined by the particular scenarios of s_n^j and the rate of regular contribution $\mathbb{T}$ by the relationship:

$$\mu_{max} = \underset{y}{Max} \quad \sum_{m \in T} p_m (y_m^{\mathbb{T}} 1) \qquad \ldots(33)$$

Subject to constraints *(23)* & *(31)* where $y_n \geq 0$ for all $n \in N$. From *(33)*, if $\mu \leq \mu_{max}$ then the optimization problems *(29)* to *(32)* must be bounded. To lower the number of variables, we assume that the investors do not make decisions about the fund selection every year but only years $0=t_0<t_1<\ldots<t_w$ represented by the tree stage *0,1,…,w*. In *figure 2*, it is observed that the last time stage of the tree correspond to real time T but this time is the time of retirement where no more decisions about investments are to be made and therefore the last decision time is $t_w<T$. Hence the depth of the tree is $w+1$.



If $l_k = t_k - t_{k-1}$ denotes the length of the period $[t_{k-1}, t_k]$, $k \in \{1,\ldots,w+1\}$, the basic problems *(22)* to *(24)* and their equivalent linear counterpart *(25)* to *(28)* are based on the assumption that l_k=1 for all k. We now assume that $l_k>1$ for all or at least some k. the retirement saving account is appreciated l_k times during the period $[t_{k-1}, t_k]$. The regular contribution $\mathbb{T}$ is transferred to the account l_k times too. It is assumed that it is distributed each time between funds $j \in \{1,...,J\}$ maintaining their weights from the previous decision time t_{k-1}. If $\mathbb{T}_n=[\mathbb{T}_n^1, \ldots, \mathbb{T}_n^J]^{\mathbb{T}}$ is the vector of yearly contributions transferred $l_{\varepsilon\{n\}+1}$ times to fund $\{1,...,J\}$ during the period $[t_{\varepsilon\{n\}}, t_{\varepsilon\{n\}+1}]$ from the node n to any of its successors from the set $\{n\}^+$, $n \in N \setminus T$, then:

$$\frac{\mathbb{T}_n^J}{\mathbb{T}} = \frac{y_n^J}{y_n^{\mathbb{T}}1} \quad ...(34)$$

Where $\mathbb{T}_n^{\mathbb{T}}1 = \mathbb{T}$ for all $n \in N \setminus T$. Constraints *(23) & (28)*, concerning the appreciation of the wealth, have to be modified in the way below.

$$y_n^{\mathbb{T}}1 = y_{n\_}^{\mathbb{T}} s_n + \mathbb{T}_{n\_}^{\mathbb{T}} \sum_{i=0}^{l_{\varepsilon\{n\}-1}} (s_n)^{\frac{i}{l_{\varepsilon\{n\}}}} \text{ for all } n \in N_0 \quad ...(35)$$

And $y_n \geq 0$ for all $n \in N$.

The components of the vector $\mathbb{T}_{n\_}$ are given by *(34)*. The power $(s_n)^{\frac{i}{l_{\varepsilon\{n\}}}}$ of the vector of yearly adjusted returns is considered component wise, *i.e*, $(s_n)^{\frac{i}{l_{\varepsilon\{n\}}}} = [(s_n^1)^{\frac{i}{l_{\varepsilon\{n\}}}}, \ldots, (s_n^J)^{\frac{i}{l_{\varepsilon\{n\}}}}]^{\mathbb{T}}$.

4.0 ALGORITHM AND RESULT.

We consider the risk model minimizing *(25)* subject to *(24), (28), (34)* and *(35)*. Problem *(25)* can be solved iteratively in the following way.

1. Fix the starting point $\mathbb{T}_n^J = \mathbb{T}/J$ for all $n \in N \setminus \mathbb{T}$, $j \in \{1,...,J\}$
2. Solve the problem *(25)* subject to *(24), (28), (34)* and *(35)* with the fixed parameter $\mathbb{T}_n^J$
3. Obtain optimal y_n^J for all n, j
4. Compute new $\mathbb{T}_n^J$ for all n, j using *(34)*
5. Repeat steps until desired accuracy of iterates is attained.

If we denote the solution of the kth iterate by $x^{(k)} = [a^{(k)}, z^{(k)}, y^{(k)}]$, then the optimization problem(in line with *(29)*) solved by the algorithm is:

$$\underset{x}{Min} \; c^{\mathbb{T}} x^{(k+1)} \quad ...(36)$$

Subject to:

$$A_{ineq}\, x^{(k+1)} \leq b_{ineq}, \quad ...(37)$$
$$A_{eq} x^{(k+1)} = b_{eq}\,(x^{(k)}), \quad ...(38)$$
$$Z^{(k+1)}, y^{(k+1)} \geq 0 \quad ...(39)$$

Where the right hand side vector from the equality constraints has elements

$$[\, b_{eq}\,(x^{(k)}]_i = [\, b_{eq}\,(y_{n\_}^{(k)})]_i = \begin{cases} \mathbb{T} & i = 1 \\ \mathbb{T}\,\dfrac{(y_{n\_}^{(k)})^{\mathbb{T}} q_n}{(y_{n\_}^{(k)})^{\mathbb{T}}\, 1} & i = 2, \dots, S+1,\ \ where\ \ n = i-1 \\ 0 & i = N-S+2, \dots, N+1, \end{cases}$$

where $q_n = \sum_{i=0}^{l_{\xi\{n\}}-1} (s_n)^{\frac{i}{l_{\xi\{n\}}}}$ as pointed out in *(35).*

From *(18),* it is worthwhile to note that any choice of the family H_t, $t=0,\dots,T$, of positive bounded scaling functions does not change the result. It may, however, improve the stability of numerical computation. In order to capture both large and small values of V_t, we recursively define the scaling functions H_t, $t=T, T-1,\dots,1, 0$, depending on the computed V_{t+1} as follows:

$$H_T = \frac{1}{\sqrt{1+V_T^2}} \quad \text{and} \quad H_t = \frac{1}{\sqrt{1+V_{t+1}^2}} \quad \text{for } t=T-1,\dots,0$$

In our algorithm, we compute values of the function $W_t = W_t(d)$ for discrete values of d from the time dependent interval $d\epsilon(d_{min}, {}^{t}/_{2})$, where $d_{min}=d_0=0.08$ (employee's contribution rate). The upper bound ${}^{t}/_{2}$ has been chosen with respect to maximal expected values of the savings/contributions to salary ratio d. Stochastic fund returns r_t^j are assured to have normal distributions with densities f_t^j having constant means $\tilde{r}^j$ and standard deviation δ^j, $j=1,\dots,J$.

PORTFOLIO COMPOSITION

Table 1 has presented the fund types and the maximum/minimum portions of the fund exposed to investment risk as contained in the guidelines issued by the Nigerian National Pension Commission (PenCom). In line with this, let us now present more detailed characteristics of the four funds. The funds majorly invest into stocks, real estate, bonds and other money market instruments. For simplicity, let us categorize the investments into stock(**S**) and bonds (**B**) as risky and less risky respectively. Therefore, *figure 3* depicts the investment portfolio having categorized the investment assets listed in the Section *86* (Part *XII*) of PenCom Guidelines as stock and bonds.

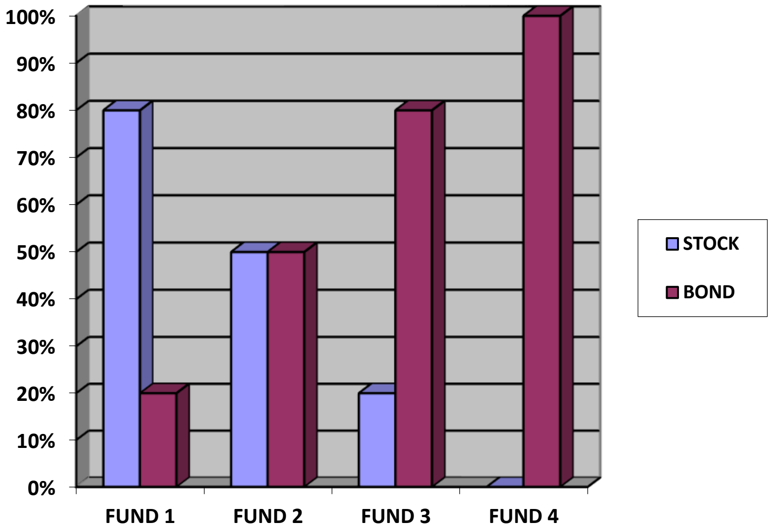


Figure 3: PENSION FUND INVESTMENT PORTFOLIO COMPOSITION

Given that the average returns on stock ($\tilde{r}^s$) and bond ($\tilde{r}^b$) are respectively *0.12345* and *0.06789,* and the equivalent standard deviations are *0.20419*(δ^s) and *0.04535*(δ^b) respectively. According to *Table 1* and under the assumption that the pension fund management institutions use the maximal possible extent, we may express the values of the funds symbolically and mathematically in the following ways:

FUND I = *0.8**S** + 0.2**B***
FUND 2 = *0.5**S** + 0.5**B***
FUND 3 = *0.2**S** + 0.8**B***
FUND 4 = 1.0***B***

In diagrammatic term;

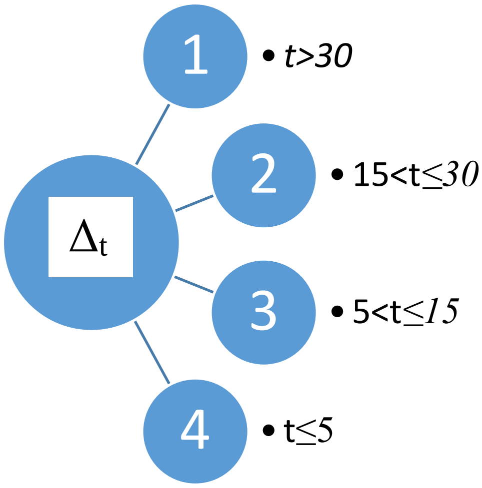


Figure 4: **FUND**

Where Δ_t is the corresponding fund type which depends on the time parameter t and the restrictions/constraints $j(t, d) \in \Delta_t$ for all $t=1,\ldots,T$. t here is the time (in years) until retirement. In other words, $\Delta_t \subset \{1,\ldots,J\}$ represents the set of all funds that may be chosen by an investor at time t having taken into account the government restrictions imposed on such fund selection. By simple a computation, *Table 2* shows the average returns and standard deviations of the funds. We assume the period of saving is $T=35$ while the percentage of an employee's salary transferred to the RSA is $\mathbb{T}=8\%$.

j	$\tilde{r}^s$	$\tilde{r}^b$	δ^s	δ^b	$\tilde{r}^j$	δ^j
1	0.12345	0.06789	0.20419	0.04535	0.112338	0.172422
2	0.12345	0.06789	0.20419	0.04535	0.09567	0.12477
3	0.12345	0.06789	0.20419	0.04535	0.079002	0.077118
4	0.12345	0.06789	0.20419	0.04535	0.06789	0.04535

*Table 2: **AVERAGE RETURNS AND DEVIATION***

In determining the optimal choice $j=j(d, t)$ of a fund depending on time $t \in \{0,\ldots,T-1\}$ and the average saved money to wage ratio , one can observe that the optimal strategy is to choose the most risky fund in the early years of saving in order to maximize wealth and gradually switch to less risky funds in later years in order to ensure safety of such wealth at retirement. Therefore, special attention is paid to the behavior of the optimal strategy j and the saved amount $E(d_T)$ when the risk aversion parameter a entering the utility function changes. It can be observed that increasing the aversion to risk causes that the switching times between funds move to the earlier times. That is, the investors switch from Fund *1* to Fund *2* earlier as well as

Fund *2* to Fund *3* or from Fund *3* to Fund *4*. Invariably, for higher values of the risk aversion parameters, one typically obtains lower levels of the final wealth. Although, it must be noted that higher exposure to risky assets for low risk aversion may also result in low outcome in case of unfavourable behaviour of stock market. But in general, lowering risk aversion results to a higher outcome $E(d_T)$ on the average. Under this assumption of average returns $\tilde{r}^j$ and volatilities δ^j of stock and bond, the optimal pension investment fund strategy for wealth maximization model is as follows:

- Decreasing in time t and also decreasing in d_t
- Decreasing in risk aversion parameter a
- Increasing in average stock returns $\tilde{r}^s$
- Decreasing in average bond returns $\tilde{r}^b$

On the other hand, the optimal strategy for risk minimization model is as follows:

- Increasing in time t
- Increasing in risk aversion parameter a
- Decreasing in stock returns $\tilde{r}^s$
- Increasing in average bond returns $\tilde{r}^b$

5.0 SUMMARY, CONCLUSION AND RECOMMENDATION.

Pension is one aspect that needs efficient and effective management because of the funds involved as the living standard of the retirees depends on it after retirement. The major aspects that require special intervention in the administration and management of pension fund assets are risk minimization and return maximization. These two areas are very delicate because there is existence of a direct relationship between them. That is, any attempt geared towards increasing the wealth of a proposed retiree through the investments of pension fund assets will also bring about the increase in risk of losing same funds if investment planning and strategy are not adequately put in place by the investors of funds. In other words, the higher the returns (wealth), the higher will be the corresponding risk (uncertainty) and vice versa. Therefore, there is a need to strike the balance between the two areas (risk and returns). In this research, two types of models were proposed for the problem of optimal fund selection in funded scheme of pension fund investment using Nigerian context. The research has been able to create a path towards achieving this solution by developing models of wealth maximization and risk minimization. Dynamic Accumulation Model (DAM) has been modified to maximize the wealth of the retirement saving account holders. The model leads to a Bellman equation of stochastic dynamic programming as shown in *(12)*. Also, Risk Minimizing Model (RMM), based on the opposite approach, ensures that the riskiness of the investment of pension fund assets is minimized and measured by the single period average value-at-risk deviation. It was also formulated on a scenario tree with adjusted fund returns as the underlying process. In $(25) - (28)$, it was shown that the model can be rewritten to a linear program. $(29) - (32)$ show that it attains an optimum feasible value of the μ parameter. The decisions about the fund selection are up to the investors subject to some constraints. Moreover, the contributors can also influence the investment decision based on a strict formal request having considered the government restrictions that apply to such request or choice of investment fund selection. The intention of the restrictions and government regulations is to help lower the risk of the value of savings falling substantially, shortly before retirement. Furthermore, the idea of switching between fund *1, 2, 3* or *4* to the less risky fund as the retirement age approaches helps to safeguard the retirement fund. The study gives experience about the quantitative character of optimal fund selection strategy. We noticed that quantitative and even qualitative properties depend on the average returns and volatilities of stock and bond that enter into the models. The average stock return is higher than the average bond return but the volatility of stock return is higher than that of bond return. One can observe that in most risky fund, *Fund 1,* the share of stock decreasing in time is in accordance to expectation because a higher amount of investible fund is more sensitive to change in the level of fund returns in order to lower the risk in future. In *(8)* and *(20),* share of stock in the investment increases when the target wealth μ increases. Moreover, if the volatility of stock returns is significantly lower than the volatility of bond returns, one can eventually expect a trend of lowering the weight of bond and raising that of stock over time. Also in *(5)* and *(6),* from the description of the value-at-risk VaR_α , higher α implies a higher VaR_α , thereby a higher average value-at-risk $AVaR_\alpha$ and hence a lower $AVaRD_\alpha$, where $AVaRD_\alpha = E - AvaR\alpha$. Therefore, a higher α leads to an investment strategy with a higher proportion of stock on the average. Because a higher amount of saved amount $\underset{\sim}{E}(d_T)$ is more sensitive to changes in the fund returns, this research work recommends that secured funds are more preferable to funds with higher volatility of returns in later times. For instance, if the average stock return is higher than the average bond returns, then the volatility of the stock returns will be higher than the volatility of bond returns. This makes the investment on the stock more risky than the investment on bond. Invariably, the investors of pension assets/fund, for safety onus, will decrease the proportion of risky assets/fund in the optimal strategy over time (i.e., as retirement time draws nearer). Finally, there is considerable room for further research on this area of study. In DAM and TRMM, we dealt with a case of future pensioners who are interested in their terminal wealth at time T of retirement only. Evolution of their account at intermediate times was not considered. There will be need to develop additional models that will incorporate multi-period risk minimization and proportional investment allocation which will consider contributors who are interested in their wealth value throughout their whole period(i.e., at any time) of savings. This will be necessary; especially in case of early death of a future pensioner whose pension wealth becomes a subject of inheritance. Furthermore, further work on this area will be certainly warranted to fully apply or implement the models modified in this research for pension fund investment in the entire Federal Republic of Nigeria (FRN). Under normal scenario, there exist changes in risk aversion parameters, stock returns, bond returns and salary growth rate. There is need to investigate the sensitivity of the optimal solution with respect to these changes also. The investible funds differ in the weights of assets in their investment allocation even if they invest in the same set of assets or financial instruments. Additional model will help to find an optimal weight of stocks in the investment strategy over time. Nevertheless, this research has set a foundation for an experience about the quantitative and qualitative character of optimal fund selection strategies. It has also set a foundation on how the optimal strategy and the amounts of contributions change in values under some parameters such as average returns and volatility of stocks and bonds.

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Cite this article:

Author(s), ABERE, OMOTAYOJOHNCALLY, ABIOLA, BANKOLE **(2019).** "On Mathematical Models for Pension Fund Optimal Selection Strategies". Name of the Journal: International Journal of Academic Research in Business, Arts and Science, (IJARBAS.COM), P, 72- 92. DOI: 10.5281/zenodo.3603511, Issue: 4, Vol.: 1, Article: 5, Month: January, Year: 2020. Retrieved from https://www.ijarbas.com/all-issues/current-articles/

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